Technical Appendix for:
Spatial Correlation Robust Inference with Errors in Location or Distance

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Our model assumes a population of agents residing at d-dimensional integer lattice locations with one individual per location. We focus on an expectation zero process $X_{\mathbf{s}}$ indexed on this lattice that is assumed to be mixing as detailed below. For simplicity, we also assume the process is stationary: the joint distribution of $X_{\mathbf{s}}$ for a collection of locations is invariant to translation and so, assuming second moments exist, $E\{X_{\mathbf{s}}X_{\mathbf{s+h}}\} = C(\mathbf{h})$. The econometrician's sample consists of realizations of agents' random variables $X_{\mathbf{s}}$ at a collection of locations $\{\mathbf{s}_i\}$ inside a sample region Λ_τ and measurements of these locations. We use the notation $|\Lambda_\tau|$ to denote the number of agents in our sample region and, for simplicity, assume that all locations in Λ_τ are sampled. When taking limits, we view Λ_τ as one of a sequence of regions indexed by τ that grow to include the whole lattice, an increasing domain approach to asymptotic approximations.

In what follows, we state the notion of mixing coefficients used throughout this Appendix, and provide proofs of the results in Conley and Molinari (2005).

1 Bolthausen CLT

Letting $\pi(s_1, s_2)$ denote maximum across coordinates of $|s_1 - s_2|$, define a distance measure between sets $\pi(\Lambda_1, \Lambda_2) = \inf\{\pi(s_1, s_2) : s_1 \in \Lambda_1, s_2 \in \Lambda_2\}$. For a mean zero stationary random vector X_s $s \in Z^d$, let F_Λ denote the sigma algebra generated by X_s, $s \in \Lambda$, $\Lambda \subset Z^d$. Define mixing coefficients as:

$$\begin{aligned}
\alpha_{k,l}(n) &= \sup\{|P(a_1 \cap a_2) - P(a_1)P(a_2)| : a_1 \in F_{\Lambda_1}, a_2 \in F_{\Lambda_2}, |\Lambda_1| \leq k, |\Lambda_2| \leq l, \pi(\Lambda_1, \Lambda_2) \geq n\} \\
\rho(n) &= \sup\{|cov(b_1, b_2)| : b_1 \in L_2(F_{\Lambda_1}), b_2 \in L_2(F_{\Lambda_2}), \|b_1\|_2 \leq 1, \|b_2\|_2 \leq 1, \pi(\Lambda_1, \Lambda_2) \geq n\}
\end{aligned}$$

Theorem 1 *(Bolthausen 1982)*
If $\sum_{m=1}^{\infty} m^{d-1}\alpha_{k,l}(m) < \infty, k+l \leq 4,\ \alpha_{1,\infty}(m) = o(m^{-d})$ *and if*

$$\sum_{m=1}^{\infty} m^{d-1}\rho(m) < \infty$$

or

$$\text{for some } \delta > 0, \|X_{\mathbf{s}}\|_{2+\delta} < \infty \text{ and } \sum_{m=1}^{\infty} m^{d-1}\alpha_{1,1}(m)^{\delta/(2+\delta)} < \infty,$$

then $\sum_{\mathbf{s}\in Z^d} |cov(X_{\mathbf{0}}, X_{\mathbf{s}})| < \infty$. *If additionally* $\sigma^2 = \sum_{\mathbf{s}\in Z^d} cov(X_{\mathbf{0}}, X_{\mathbf{s}}) > 0$, *and* Λ_τ *is a fixed sequence of finite subsets of* Z^d *that increases to* Z^d *and is such that*

$$\lim_{\tau\to\infty} |boundary(\Lambda_\tau)|/|\Lambda_\tau| = 0,$$

Then

$$\frac{1}{\sigma|\Lambda_\tau|^{1/2}} \sum_{\mathbf{s}\in\Lambda_\tau} X_{\mathbf{s}} \Rightarrow N(0,1)$$

2 Proofs of Propositions

2.1 Proposition 2

Proposition 2 *Suppose errors in locations are bounded and: (a)* $X_{\mathbf{s}}$ *is a stationary, mixing process with* $(4+\delta)$ *th moments,* $\delta > 0$, *and with alpha mixing coefficient s.t.* $\alpha_{\infty,\infty}(m)^{\delta/(2+\delta)} = o(m^{-4})$; *(b) Each* $L_{i,\tau} = o(N_{i,\tau}^{1/3})$, *and* $K(\cdot)$ *is a continuous bounded function on* $[-1,1]^2$ *with* $K(0,0) = 1$, *and such that either (i)* $K(\cdot)$ *has absolutely summable Fourier coefficients, or (ii)* $K(\cdot)$ *is a uniform function on* $[-1,1]^2$. *Then* $\hat{V}_{NP} \xrightarrow{p} V$.

Proof. The proposition using conditions (a) and (b-i) is Proposition 5 in Conley (1999). Here we provide a proof using (b-ii).

The strategy for proving consistency in the presence of bounded measurement errors in location can be cast in terms of showing that $\hat{V}_{NP}$, obtained using the uniform kernel with cutoff L and mismeasured locations, is asymptotically equivalent to an infeasible estimator that uses true locations and a smaller cutoff point.

It will be convenient in this proof to explicitly refer to each coordinate of $\mathbf{s} = (m,n)$, let the sample region Λ be an M by N rectangle, suppressing the index τ. Let the bound on measurement error in each dimension be denoted B so that for each point $|m^{true} - m^{measured}| < B$ and $|n^{ntrue} - n^{measured}| < B$. We index points throughout this proof with their true indexes. The kernel weight for the product of points

(m,n) and $(m+j,n+k)$ is denoted $\tilde{K}_{MN}(m,n,j,k)$. These weights will be zero and one, but depend on the measurement errors at both locations (m,n) and $(m+j,n+k)$.

$$\hat{V}_{NP} = \tfrac{2}{MN}\textstyle\sum_{j=0}^{L_M+2B}\sum_{k=0}^{L_N+2B}\sum_{m=j+1}^{M}\sum_{n=k+1}^{N}\tilde{K}_{MN}(m,n,j,k)X_{m,n}X_{m-j,n-k} \\ -\tfrac{1}{MN}\textstyle\sum_{m=1}^{M}\sum_{n=1}^{N}X_{m,n}^2$$

Define $\tilde{V}$ as the infeasible, consistent estimator with displacements that are small enough that they still get weight one:

$$\tilde{V} = \tfrac{2}{MN}\textstyle\sum_{j=0}^{L_M-2B}\sum_{k=0}^{L_N-2B}\sum_{m=j+1}^{M}\sum_{n=k+1}^{N}X_{m,n}X_{m-j,n-k} \\ -\tfrac{1}{MN}\textstyle\sum_{m=1}^{M}\sum_{n=1}^{N}X_{m,n}^2$$

Conley (1999), Proposition 3 directly implies that $\tilde{V}\to V$ in probability. Therefore, it suffices to show here that the difference R between $\hat{V}_{NP}$ and $\tilde{V}$ vanishes. Let $R\equiv\hat{V}_{NP}-\tilde{V}$:

$$R = \frac{2}{MN}\sum_{j=0}^{L_M+2B}\sum_{k=0}^{L_N+2B}\sum_{m=j+1}^{M}\sum_{n=k+1}^{N}[1-1(j<L_M-2B)1(k<L_N-2B)]\tilde{K}_{MN}(m,n,j,k)X_{m,n}X_{m-j,n-k}$$

The result follows from a demonstration that $R\to 0$ in mean square. $ER=0$ for L_N, L_M large enough since $X_{m,n}$ is a finite-order moving average, so showing $var(R)\to 0$ is sufficient. We first show $E(R-ER)^2\to 0$ and then $ER\to 0$.

To simplify notation let $X_{m,n}=0$ for non-positive values of either index. Define an array $Z_{MN,mn}$:

$$Z_{MN,mn} = \\ \sum_{j=0}^{L_M+2B}\sum_{k=0}^{L_N+2B}[1-1(j<L_M-2B)1(k<L_N-2B)]\tilde{K}_{MN}(m,n,j,k)(X_{m,n}X_{m-j,n-k}-EX_{m,n}X_{m-j,n-k}),$$

so $R-ER=\frac{2}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}Z_{MN,mn}$. Hence,

$$var\{\sum_{m=1}^{M}\sum_{n=1}^{N}Z_{MN,mn}\} = \left|\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{m'=1}^{M}\sum_{n'=1}^{N}EZ_{MN,mn}Z_{MN,m'n'}\right|.$$

The triangle inequality implies:

$$var\{\sum_{m=1}^{M}\sum_{n=1}^{N}Z_{MN,mn}\} \le \left\{\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{|m-m'|\le 2(L_M+2B)}\sum_{|n-n'|\le 2(L_N+2B)}\left|EZ_{MN,mn}Z_{MN,m'n'}\right| + \\ \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{m',n':|m-m'|>2(L_M+2B)\text{or}|n-n'|>2(L_N+2B)}\left|EZ_{MN,mn}Z_{MN,m'n'}\right|\right\}, \tag{1}$$

having divided the terms into close ones (within $2(L_M+2B)$ and $2(L_N+2B)$ in each direction) and far ones (farther than $2(L_M+2B)$ or $2(L_N+2B)$). Note that if the sample region were not rectangular, the $EZ_{MN,mn}Z_{MN,m'n'}$ terms could still be divided into close and far groups of terms.

First look at the close terms. No matter what the shape of the sample region, the maximum number of points within $2(L_M+2B)$, $2(L_N+2B)$ in each direction from any point is $(4L_M+8B+1)(4L_N+8B+1)$. Therefore:

$$\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{|m-m'|<2(L_M+2B)}\sum_{|n-n'|<2(L_N+2B)}\left|EZ_{MN,mn}Z_{MN,m'n'}\right| \le$$

$$MN(4L_M+8B+1)(4L_N+8B+1)\sup_{1\le m'\le M,1\le n'\le N}\left\|Z_{MN,m'n'}\right\|_2^2.$$

The next step is to bound $\sup_{1\le m'\le M,1\le n'\le N}\left\|Z_{MN,m'n'}\right\|_2^2$. Minkowski's inequality implies:

$$\begin{aligned} &\|Z_{MN,mn}\|_2 \le \\ &\left\{\sum_{j=0}^{L_M+2B}\sum_{k=0}^{L_N+2B}[1-1(j<L_M-2B)1(k<L_N-2B)]\cdot\ldots\right. \\ &\left.\tilde{K}_{MN}(m,n,j,k)\left\|X_{m,n}X_{m-j,n-k}-EX_{m,n}X_{m-j,n-k}\right\|_2\right\}. \end{aligned} \tag{2}$$

$X_{m,n}$ has finite $(4+\delta)th$ moments which implies that $\sup_{j,k}\left\|X_{m,n}X_{m-j,n-k}-EX_{m,n}X_{m-j,n-k}\right\|_2$ is bounded, $\tilde{K}_{MN}(m,n,j,k)$ are uniformly bounded, and the number of terms where $[1-1(j<L_M-2B)1(k<L_N-2B)]=1$ is $[(L_M+2B+1)(L_N+2B+1)-(L_M-2B+1)(L_N-2B+1)]=4B(L_M+L_N+2)$. Hence

$$\|Z_{MN,mn}\|_2 \le c_1(L_M+L_N+2)$$

for some constant c_1. Thus giving the following bounds:

$$\|Z_{MN,mn}\|_2^2 \le c_1^2(L_M+L_N+2)^2$$

Therefore the near terms satisfy:

$$\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{|m-m'|<2L_M}\sum_{|n-n'|<2L_N}\left|EZ_{MN,mn}Z_{MN,m'n'}\right| \le c_1^2MN(4L_M+8B+1)(4L_N+8B+1)(L_M+L_N+2)^2. \tag{3}$$

Next consider the far apart terms.

$$\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{m',n':|m-m'|>2L_M \text{or} |n-n'|>2L_N}\left|EZ_{MN,mn}Z_{MN,m'n'}\right|$$

A mixing inequality from Ibragimov and Linnik (1971) chapter 17 gives a bound on $\left|EZ_{MN,mn},Z_{MN,m'n'}\right|$:

$$\left|EZ_{MN,mn},Z_{MN,m'n'}\right| \le c_2\alpha_{\infty,\infty}(\min(2(L_M+2B),2(L_N+2B))^{\delta/(2+\delta)}\sup_{m,n}\|Z_{MN,mn}\|_{2+\delta}^2$$

and an argument identical to that above for (2) implies $\|Z_{MN,mn}\|_{2+\delta}^2 \le c_3^2(L_M+L_N+2)^2$ for some constants c_2, c_3. Combining these terms give a bound on the far terms of:

$$\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{m',n':|m-m'|>2L_M \text{or} |n-n'|>2L_N}\left|EZ_{MN,mn}Z_{MN,m'n'}\right| \leq$$
$$c_4 M^2 N^2 \alpha_{\infty,\infty}(\min(2(L_M+2B), 2(L_N+2B))^{\delta/(2+\delta)}(L_M+L_N+2)^2$$

Combining the bounds on near and far terms yields:

$$var\{\sum_{m=1}^{M}\sum_{n=1}^{N} Z_{MN,mn}\} \leq c_1^2 MN(4L_M+8B+1)(4L_N+8B+1)(L_M+L_N+2)^2+$$
$$c_4 M^2 N^2 \alpha_{\infty,\infty}(\min(2(L_M+2B), 2(L_N+2B))^{\delta/(2+\delta)}(L_M+L_N+2)^2 + o(1).$$

The rate conditions on $L_{i,\tau}$ and the mixing condition in parts (a) and (c) imply that the right side of this expression converges to zero as $M, N \to \infty$.

Consider now ER :

$$ER =$$
$$= \tfrac{2}{MN}\sum_{j=0}^{L_M+2B}\sum_{k=0}^{L_N+2B}\sum_{m=j+1}^{M}\sum_{n=k+1}^{N}[1-1(j<L_M-2B)1(k<L_N-2B)]\tilde{K}_{MN}(m,n,j,k)EX_{m,n}X_{m-j,n-k}$$
$$= 2\sum_{j=0}^{L_M+2B}\sum_{k=0}^{L_N+2B}[1-1(j<L_M-2B)1(k<L_N-2B)]\tilde{K}_{MN}(m,n,j,k)\tfrac{(M-j)(N-k)}{MN}EX_{m,n}X_{m-j,n-k}$$

Using the same mixing inequality to bound $EX_{m,n}X_{m-j,n-k}$:

$$|ER| \ \leq$$
$$c_5\sum_{j=0}^{L_M+2B}\sum_{k=0}^{L_N+2B}\left|[1-1(j<L_M-2B)1(k<L_N-2B)]\tilde{K}_{MN}(m,n,j,k)\tfrac{(M-j)(N-k)}{MN}\right|\alpha_{\infty,\infty}(\max(j,k))^{\frac{\delta}{(2+\delta)}}$$
$$\leq c_5\sum_{j=0}^{L_M+2B}\sum_{k=0}^{L_N+2B}\left|[1-1(j<L_M-2B)1(k<L_N-2B)]\right|\alpha_{\infty,\infty}(\max(j,k))^{\frac{\delta}{(2+\delta)}}.$$

$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha_{\infty,\infty}(\max(j,k))^{\delta/(2+\delta)} < \infty$ since $\alpha_{\infty,\infty} = o(m^{-4})$, so the dominated convergence theorem implies

$$c_5\sum_{j=0}^{L_M+2B}\sum_{k=0}^{L_N+2B}|[1-1(j<L_M-2B)1(k<L_N-2B)]|\alpha_{\infty,\infty}(\max(j,k))^{\delta/(2+\delta)} \to 0$$

since $|[1-1(j<L_M-2B)1(k<L_N-2B)]| \to 0$, all j,k. ■

2.2 Proposition 3

Proposition 3 *For correctly measured distances, the Gaussian DGPs and sampling framework for the* $\{X_{\mathbf{s}}\}$ *process satisfy the conditions in Theorem 3 of Mardia and Marshall (1984). Hence the MLE estimator is consistent and asymptotically normal.*

Notation

We first introduce some notation that will be used in the remaining proofs. Let Λ_τ be the hypercube (in $d-$dimensional Euclidean space) of lattice points $\mathbf{s}$ with all components integers s_i, $1 \leq s_i \leq N$, so that $N^d = |\Lambda_\tau|$. Given our DGP, $\{X_{\mathbf{s}}\}$ is a random field with $EX_{\mathbf{s}} \equiv 0$ and cumulant functions up to order eight absolutely summable. Denote by

$$f(\boldsymbol{\omega}) \equiv \frac{1}{(2\pi)^d} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_d=-\infty}^{\infty} r_{\mathbf{s}} e^{-i\mathbf{s}\cdot\boldsymbol{\omega}}$$

the spectral density of $\{X_{\mathbf{s}}\}$, where $r_{\mathbf{s}} \equiv C(\mathbf{s}) = E(X_{\mathbf{s}} X_{\mathbf{u}+\mathbf{s}})$, $\omega = (\omega_1, \ldots, \omega_d) \in [-\pi, \pi]^d$, and $(\mathbf{s} \cdot \boldsymbol{\omega}) = \sum_{h=1}^{d} s_h \omega_h$ is the inner product in $d-$dimensional Euclidean space. For locations on the plane, $d = 2$. Let Σ denote the variance-covariance matrix of the vector $[X_{\mathbf{s}}, \mathbf{s} \in \Lambda_\tau]$. Let

$$\begin{aligned} \Sigma_\rho &= \frac{\partial \Sigma}{\partial \rho}, \quad \Sigma_{\sigma^2} = \frac{\partial \Sigma}{\partial \sigma^2}, \\ \vartheta_{ij} &= \operatorname{tr}\left(\Sigma^{-1}\Sigma_i \Sigma^{-1}\Sigma_j\right), \quad i, j = \rho, \sigma^2. \end{aligned}$$

Let $\theta = (\rho, \sigma^2)$ denote the true values of the parameters in the DGP. Let $\hat{\theta}_{MLE} = (\hat{\rho}_{MLE}, \hat{\sigma}^2_{MLE})$ denote the MLE estimator of θ.

Proof.

Mardia and Marshall (1984, Theorem 3) show that $\hat{\theta}_{MLE}$ is consistent and asymptotically normal, provided that $C(\mathbf{k}; \rho, \sigma^2)$ and its first and second derivatives are absolutely summable, and that

$$\begin{aligned} a_{ij} &= \lim \frac{\vartheta_{ij}}{(\vartheta_{ii}\vartheta_{jj})^{\frac{1}{2}}} \ \text{ exists}, \ \ i, j = (\rho, \sigma^2), \\ \det(A) &= \det\left(\begin{bmatrix} a_{\rho\rho} & a_{\rho\sigma^2} \\ a_{\rho\sigma^2} & a_{\sigma^2\sigma^2} \end{bmatrix}\right) \neq 0. \end{aligned}$$

The covariance functions $C(\mathbf{k}; \rho, \sigma^2)$ are polynomials in ρ and σ^2 for each $\mathbf{k} \in \Lambda_\tau$, and therefore their derivatives exist and are continuous. Absolute summability is ensured by the fact that the processes we consider are finite order moving averages, and therefore

$$\sum_{\mathbf{k} \in Z^2} \left|C(\mathbf{k}; \rho, \sigma^2)\right| = \sum_{\mathbf{k} \in Z^2 : \|\mathbf{k}\| \leq 3} \left|C(\mathbf{k}; \rho, \sigma^2)\right| < \infty,$$

with similar considerations for the first and second derivatives of $C(\mathbf{k}; \rho, \sigma^2)$.

From the above considerations it follows that there exists a positive finite constant η_1 such that

$$\sum_{\mathbf{k}\in Z^2}(1+|k_l|)\left|\Delta C\left(\mathbf{k};\rho,\sigma^2\right)\right|<\eta_1,\quad l=1,2$$

where k_l denotes the $l-$th component of $\mathbf{k}$, and Δ denotes either the identity operator 1, one of the first order differential operators $\partial/\partial\theta_i$, or one of the second order operators $\partial^2/\partial\theta_i\partial\theta_j$ $i,j=1,2$. Hence, $\left|(2\pi)^2\Delta f(\boldsymbol{\omega})\right|<\eta_1$. Moreover, given our choice of an MA(6) process, and our values of $\rho=0.3$, 0.45, it follows that there exists a positive finite constant η_2 such that:

$$\frac{1}{(2\pi)^2 f(\boldsymbol{\omega})}<\eta_2.$$

This implies that the conditions for Theorem 1 and Lemmas 3.1-3.2, 4.1-4.3 in Kent and Mardia (1996) are satisfied, and therefore

$$\operatorname{tr}\left(\Sigma^{-1}\Sigma_i\Sigma^{-1}\Sigma_j\right)=\frac{|\Lambda_\tau|}{(2\pi)^2}\int\frac{f_i(\boldsymbol{\omega})f_j(\boldsymbol{\omega})}{f(\boldsymbol{\omega})^2}d\boldsymbol{\omega}+O\left(\sqrt{|\Lambda_\tau|}\right)$$

where $f_i(\boldsymbol{\omega})=\frac{\partial f(\boldsymbol{\omega})}{\partial\theta_i}$. Hence, for $i,j=1,2$

$$\begin{aligned}\lim_{|\Lambda_\tau|\to\infty}\frac{\vartheta_{ij}}{\sqrt{(\vartheta_{ii}\vartheta_{jj})}} &= \lim_{|\Lambda_\tau|\to\infty}\frac{\frac{|\Lambda_\tau|}{(2\pi)^2}\int\frac{f_i(\boldsymbol{\omega})f_j(\boldsymbol{\omega})}{f(\boldsymbol{\omega})^2}d\boldsymbol{\omega}+O\left(\sqrt{|\Lambda_\tau|}\right)}{\sqrt{\left(\frac{|\Lambda_\tau|}{(2\pi)^2}\int\frac{f_i(\boldsymbol{\omega})^2}{f(\boldsymbol{\omega})^2}d\boldsymbol{\omega}+O\left(\sqrt{|\Lambda_\tau|}\right)\right)\left(\frac{|\Lambda_\tau|}{(2\pi)^2}\int\frac{f_j(\boldsymbol{\omega})^2}{f(\boldsymbol{\omega})^2}d\boldsymbol{\omega}+O\left(\sqrt{|\Lambda_\tau|}\right)\right)}}\\ &= \frac{\int\frac{f_i(\boldsymbol{\omega})f_j(\boldsymbol{\omega})}{f(\boldsymbol{\omega})^2}d\boldsymbol{\omega}}{\sqrt{\left(\int\frac{f_i(\boldsymbol{\omega})^2}{f(\boldsymbol{\omega})^2}d\boldsymbol{\omega}\right)\left(\int\frac{f_j(\boldsymbol{\omega})^2}{f(\boldsymbol{\omega})^2}d\boldsymbol{\omega}\right)}}.\end{aligned}$$

The above limits exist. Given our analytic forms for the spectral densities and its derivatives with respect of ρ and σ^2, direct computations show that $\det(A)\neq 0$. ■

2.3 Proposition 4

Proposition 4 *Given our DGP for $X_{\mathbf{s}}$ on the plane and correctly measured distances, the uniform kernel function*

$$K_\tau(\mathbf{s}_i-\mathbf{s}_j)=\begin{cases}1 & \text{if } |s_{i1}-s_{j1}|\le L_\tau, |s_{i2}-s_{j2}|\le L_\tau,\\ 0 & \text{otherwise},\end{cases}\tag{4}$$

and choosing L_τ so that $L_\tau\to\infty$ and $\frac{L_\tau}{|\Lambda_\tau|^{1/2}}\to 0$ as $|\Lambda_\tau|\to\infty$, it follows that:

$$\sqrt{\frac{|\Lambda_\tau|}{L_\tau^2}}\left(\hat{V}_{NP}-\hat{V}_i\right)\xrightarrow{d}N\left(0,8V^2\right),\quad i=MLE,\ MM.$$

Notation.

We use the same notation as in the proof of Proposition 3.

Proof.

1. Asymptotic Distribution of Spectral Density Estimator on the Plane

Let $\hat{r}_{\mathbf{s}}$ denote an estimate of $r_{\mathbf{s}} \equiv C(\mathbf{s}) = E(X_{\mathbf{s}} X_{\mathbf{u}+\mathbf{s}})$ given by

$$\hat{r}_{\mathbf{s}} = \frac{1}{|\Lambda_\tau|} \sum_{\mathbf{u},\mathbf{u}+\mathbf{s} \in \Lambda_\tau} X_{\mathbf{s}} X_{\mathbf{u}+\mathbf{s}},$$

Notice that

$$E(\hat{r}_{\mathbf{s}}) = \frac{\prod_{h=1}^{d} (N - |s_h|)}{|\Lambda_\tau|} r_{\mathbf{s}},$$

An estimate $\hat{f}(\boldsymbol{\omega})$ of $f(\boldsymbol{\omega})$ is then given by

$$\hat{f}(\boldsymbol{\omega}) = \frac{1}{(2\pi)^d} \sum_{|s_1|,\ldots,|s_d| \le N} K\left(\frac{s_1}{L_1}, \frac{s_2}{L_2}, \ldots, \frac{s_d}{L_d}\right) \hat{r}_{\mathbf{s}} e^{-i\mathbf{s}\cdot\boldsymbol{\omega}},$$

where $K(\mathbf{0}) = 1$, and $K(\mathbf{x})$ is assumed to be an even $(K(\mathbf{x}) = K(-\mathbf{x}))$ function, uniformly bounded and square integrable. Given $N^d = |\Lambda_\tau|$, let $L_i \to \infty$ and $\frac{L_i}{N} \to 0$ as $N \to \infty$, $i = 1, \ldots, d$. Rosenblatt (1985) Theorem 7 p. 157 is as follows:

Theorem 5 (Rosenblatt (1985)) *Let $\{X_{\mathbf{s}}\}$ be a strictly stationary strongly mixing random field with $EX_{\mathbf{s}} \equiv 0$. Assume that the cumulant functions up to eighth order are absolutely summable. Also let the spectral density estimate $\hat{f}(\boldsymbol{\omega})$ have weights $K(\cdot)$ satisfying the condition specified above. It then follows that*

$$\sqrt{\frac{|\Lambda_\tau|}{L^d}} \left[\hat{f}(\boldsymbol{\omega}) - E\left(\hat{f}(\boldsymbol{\omega})\right)\right] \xrightarrow{d} N(0, \Omega),$$

where

$$\begin{aligned}
\Omega &= (2\pi)^d \{1 + \eta(2\omega_1) \ldots \eta(2\omega_d)\} f^2(\boldsymbol{\omega}) \int W^2(\boldsymbol{\alpha}) \, d\boldsymbol{\alpha}, \\
\eta(\mu) &= \begin{cases} 1 & \textit{if } \mu = 2m\pi, \; m \textit{ integer} \\ 0 & \textit{otherwise.} \end{cases} \\
W(\boldsymbol{\alpha}) &= \frac{1}{(2\pi)^d} \int K(\mathbf{u}) e^{-i\mathbf{u}\cdot\boldsymbol{\alpha}} d\mathbf{u}
\end{aligned}$$

□

Hence at frequency zero, under the above assumptions,

$$\sqrt{\frac{|\Lambda_\tau|}{L^d}} \left[\hat{f}(\mathbf{0}) - E\left(\hat{f}(\mathbf{0})\right)\right] \xrightarrow{d} N\left(0, (2\pi)^d \, 2 f^2(\mathbf{0}) \int W^2(\boldsymbol{\alpha}) \, d\boldsymbol{\alpha}\right).$$

Recall that $V = 2\pi f(0)$, and that we use the uniform kernel in (4). Our DGP satisfies the assumptions of Theorem 5. Additionally,

$$\begin{aligned} V &= C(0) + 4C(1) + 4C\left(\sqrt{2}\right) + 4C(2) + 8C\left(\sqrt{5}\right) + 4C\left(\sqrt{8}\right) + 4C(3) + ... \\ &\quad +8C\left(\sqrt{10}\right) + 8C\left(\sqrt{13}\right) + 4C(4) + 8C\left(\sqrt{17}\right) + 4C\left(\sqrt{18}\right) + 8C\left(\sqrt{20}\right) + ... \\ &\quad +4C([0,5]) + 8C([3,4]) + 8C\left(\sqrt{26}\right) + 8C\left(\sqrt{29}\right) + 4C\left(\sqrt{32}\right) + 4C(6) \end{aligned}$$

Therefore,

$$\left(E\left(\hat{V}_{NP}\right) - V\right) = O_p\left(|\Lambda_\tau|^{-1/2}\right)$$

Hence

$$\lim_{N\to\infty} \sqrt{\frac{|\Lambda_\tau|}{L^2}} \left[E\left(f_N(\mathbf{0})\right) - f(\mathbf{0})\right] = 0,$$

from which

$$\sqrt{\frac{|\Lambda_\tau|}{L^2}} \left[\hat{f}(\mathbf{0}) - f(\mathbf{0})\right] \xrightarrow{d} N\left(0, (2\pi)^2 2f^2(\mathbf{0}) \int W^2(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}\right). \tag{5}$$

2. Asymptotic Distribution of Specification Test on the Plane

We want to show that

$$\sqrt{\frac{|\Lambda_\tau|}{L^2}} \left(\hat{V}_{NP} - \hat{V}_i\right) \xrightarrow{d} N\left(0, \frac{1}{(2\pi)^2} 2V^2 \int \left(\int K(\mathbf{x})\, e^{-i\mathbf{x}\cdot\boldsymbol{\alpha}} d\mathbf{x}\right)^2 d\boldsymbol{\alpha}\right), \quad i = MLE, MM.$$

It is then easy to verify that with a uniform kernel $\frac{1}{(2\pi)^2} \int \left(\int K(\mathbf{x})\, e^{-i\mathbf{x}\cdot\boldsymbol{\alpha}} d\mathbf{x}\right)^2 d\boldsymbol{\alpha} = \mathbf{4}$.

Consider first the MLE estimator. The goal is to show that $\sqrt{\frac{|\Lambda_\tau|}{L^2}} \left(\hat{V}_{MLE} - V\right) \xrightarrow{p} 0$; then the desired result will follow from (5). As shown in Proposition 3, our model with $\rho = 0.3$ and $\rho = 0.45$ satisfies the conditions of Theorem 3 of Mardia and Marshall (1984). Hence

$$\sqrt{|\Lambda_\tau|} \left(\hat{\theta}_{MLE} - \theta\right) \xrightarrow{d} (0, H),$$

where H is the variance-covariance matrix of θ. Since $\hat{V}_{MLE}$ is given by the product of $\hat{\sigma}^2_{MLE}$ and a polynomial in $\hat{\rho}_{MLE}$, the desired result follows.

Consider now the MM estimator. Since our MM estimator uses unbiased covariances, Guyon's (1982) results ensure that $\sqrt{\frac{|\Lambda_\tau|}{L^2}} \left(\hat{\theta}_{MM} - \theta\right) \xrightarrow{p} 0$, from which the result follows. ■

3 Analytic Expressions for the Covariance Function and the Asymptotic Variance on the Plane

The DGP for $X_\mathbf{s}$ is

$$X_\mathbf{s} = \sum_{\mathbf{r}:\|\mathbf{s}-\mathbf{r}\|\leq 3} \rho^{\|\mathbf{s}-\mathbf{r}\|} u_{\mathbf{s}-\mathbf{r}}$$

where $u_{\mathbf{s}}$ is IID $N(0,\sigma^2)$. One can verify that:

$$\begin{aligned} V &= C(0)+4C(1)+4C\left(\sqrt{2}\right)+4C(2)+8C\left(\sqrt{5}\right)+4C\left(\sqrt{8}\right)+4C(3)+... \\ &\quad +8C\left(\sqrt{10}\right)+8C\left(\sqrt{13}\right)+4C(4)+8C\left(\sqrt{17}\right)+4C\left(\sqrt{18}\right)+8C\left(\sqrt{20}\right)+... \\ &\quad +4C([0,5])+8C([3,4])+8C\left(\sqrt{26}\right)+8C\left(\sqrt{29}\right)+4C\left(\sqrt{32}\right)+4C(6), \end{aligned}$$

and

$$C(0)=\sigma^2\left(1+4\rho^2+4\rho^{2\sqrt{2}}+4\rho^4+8\rho^{2\sqrt{5}}+4\rho^{2\sqrt{8}}+4\rho^6\right)$$

$$C(1)=\sigma^2\left(2\rho+4\rho^{1+\sqrt{2}}+2\rho^3+4\rho^{\sqrt{2}+\sqrt{5}}+4\rho^{2+\sqrt{5}}+4\rho^{\sqrt{5}+\sqrt{8}}+2\rho^5\right)$$

$$C\left(\sqrt{2}\right)=\sigma^2\left(2\rho^{\sqrt{2}}+4\rho^{1+\sqrt{5}}+2\rho^2+4\rho^{2+\sqrt{2}}+2\rho^{\sqrt{2}+\sqrt{8}}+4\rho^{3+\sqrt{5}}+2\rho^{2\sqrt{5}}\right)$$

$$C(2)=\sigma^2\left(3\rho^2+4\rho^{1+\sqrt{5}}+2\rho^4+4\rho^{2+\sqrt{8}}+2\rho^{2\sqrt{2}}+2\rho^{2\sqrt{5}}\right)$$

$$C\left(\sqrt{5}\right)=\sigma^2\left(2\rho^{\sqrt{5}}+2\rho^{1+\sqrt{2}}+2\rho^3+2\rho^{1+\sqrt{8}}+2\rho^{\sqrt{2}+\sqrt{5}}+2\rho^{2+\sqrt{5}}+2\rho^{3+\sqrt{2}}+2\rho^{3+\sqrt{8}}\right)$$

$$C\left(\sqrt{8}\right)=\sigma^2\left(4\rho^{1+\sqrt{5}}+2\rho^{\sqrt{8}}+2\rho^4+\rho^{2\sqrt{2}}+4\rho^{3+\sqrt{5}}\right)$$

$$C(3)=\sigma^2\left(4\rho^3+4\rho^{\sqrt{5}+\sqrt{2}}+4\rho^{\sqrt{5}+\sqrt{8}}\right)$$

$$C\left(\sqrt{10}\right)=\sigma^2\left(2\rho^{1+\sqrt{5}}+2\rho^{2+\sqrt{2}}+2\rho^4+2\rho^{\sqrt{8}+\sqrt{2}}+2\rho^{2\sqrt{5}}\right)$$

$$C\left(\sqrt{13}\right)=\sigma^2\left(2\rho^{1+\sqrt{8}}+2\rho^{\sqrt{2}+\sqrt{5}}+2\rho^{2+\sqrt{5}}+2\rho^5\right)$$

$$C(4)=\sigma^2\left(3\rho^4+2\rho^{2\sqrt{5}}+2\rho^{2\sqrt{8}}\right)$$

$$C\left(\sqrt{17}\right)=\sigma^2\left(2\rho^{\sqrt{8}+\sqrt{5}}+2\rho^{2+\sqrt{5}}+2\rho^{3+\sqrt{2}}\right)$$

$$C\left(\sqrt{18}\right)=\sigma^2\left(2\rho^6+2\rho^{2\sqrt{5}}+2\rho^{\sqrt{8}+\sqrt{2}}\right)$$

$$C\left(\sqrt{20}\right)=\sigma^2\left(2\rho^{2+\sqrt{8}}+2\rho^{3+\sqrt{5}}+\rho^{2\sqrt{5}}\right)$$

$$C([0\quad 5])=2\sigma^2\rho^5$$

$$C([3\quad 4])=2\sigma^2\rho^{\sqrt{5}+\sqrt{8}}$$

$$C\left(\sqrt{26}\right)=2\sigma^2\rho^{3+\sqrt{5}}$$

$$C\left(\sqrt{29}\right)=2\sigma^2\rho^{3+\sqrt{8}}$$

$$C\left(\sqrt{32}\right)=\sigma^2\rho^{2\sqrt{8}}$$

$$C(6)=\sigma^2\rho^6$$

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